Dimension of self-similar attractors. Open set condition.

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Def A map $T:(X,\rho)\to(\hat{X},\hat\rho)$ is called uniform contraction if $\exists r<1$: $\hat\rho(Tx,Ty)=r\rho(x,y)$.

Example: rotation with scaling.

Theorem: Let $(T_1,\dots,T_k)$ be a family of uniform contractions, X - complete metric space. Then $\exists!\ K\neq\emptyset$, compact, such that $K=\bigcup_{i=1}^k T_i(K)$. Moreover, for any $\vec p=(p_1,\dots p_k)$ - probability vector, $\exists\mu_{\vec p}$: $\mu_{\vec p}=\sum_{i=1}^k p_i(T_i)_*\mu_{\vec p}$ $(\mu_{\vec p}T_i^{-1})$

(True even for non-uniform contractions, i.e. just $\leq r\rho(x,y)$). K is called an attractor of the system $(T_1,\dots,T_k)$.

Examples: 1) Cantor set: 2 with ratios $\frac13$.
2) Von Koch Snowflake

4 with ratio $\frac13$

Let K be an attractor of $(T_1,\dots,T_k)$ with contraction ratios $(r_1,\dots,r_k)$ correspondingly. The self-similarity dimension of K is defined as the unique α such that $r_1^\alpha+\dots+r_k^\alpha=1$. ($\alpha$ exists and unique since $f(\alpha):=r_1^\alpha+\dots+r_k^\alpha$ is strictly increasing, $f(0)=k\geq1$, $f(\alpha)\to0$ as $\alpha\to\infty$). For our examples: Cantor set $\alpha=\frac{\log 2}{\log 3}$. Von Koch $\alpha=\frac{\log 4}{\log 3}$.

Lemma: For such a K:
1) $H_\alpha(K)=m_\alpha(K)<\infty$
2) For any m_α-measurable subset $E\subset K$, $m_\alpha(E)=H_\alpha(E)$.
3) $H_\alpha(T_i(K)\cap T_j(K))=0$, $i\neq j$.

Proof 1) We only need to prove $H_\alpha(K)\geq m_\alpha(K)$ - opposite always true.
Take any cover such that $\sum(\operatorname{diam}E_j)^\alpha\leq H_\alpha(K)+\varepsilon$. Now pick n large enough so that $(\max\operatorname{diam}E_j)(\max r_i)^n<\varepsilon$. Then $T_\sigma E_j$, where σ is a multi-index of size n, $\sigma=(\sigma_1,\dots,\sigma_n)$, and $T_\sigma:=T_{\sigma_1}\circ T_{\sigma_2}\circ\dots\circ T_{\sigma_n}$ is a cover $\supset K$ with $\max\operatorname{diam}T_\sigma E_j<\varepsilon$. So
Also $\sum_\sigma\sum_j(\operatorname{diam}T_\sigma E_j)^\alpha=\left(\sum r_i^\alpha\right)^n\sum(\operatorname{diam}E_j)^\alpha=\sum(\operatorname{diam}E_j)^\alpha$.
$m_{\alpha,\varepsilon}(K)\leq H_\alpha(K)+\varepsilon$. Let $\varepsilon\to0$.
2) Note that $H_\alpha(K)\leq H_\alpha(E)+H_\alpha(K\setminus E)\leq m_\alpha(E)+m_\alpha(K\setminus E)=m_\alpha(K)=H_\alpha(K)$ ∎
3) $m_\alpha(K)=m_\alpha\left(\bigcup T_i(K)\right)\leq\sum m_\alpha(T_i(K))=\sum r_i^\alpha m_\alpha(K)=m_\alpha(K)$.
So $m_\alpha(T_i(K)\cap T_j(K))=0$ $\forall i\neq j$ ∎

Tempting to say: $\alpha=$ Hdim K. Not always:

$T_1=\frac23x$, $T_2=\frac23x+1$. Self-similarity dimension is $2\left(\frac23\right)^\alpha=1\Rightarrow$ $\alpha=\frac{\log\frac12}{\log\frac23}>1$ - cannot be subset of $\mathbb{R}$!

Def. A family of maps $(T_1,\dots,T_k)$ satisfy Open set Condition (OSC) if $\exists$ bounded non-empty open U such that $T_i(U)\subset U$ $\forall i$ and $\forall i\neq j$ $T_i(U)\cap T_j(U)=\emptyset$.

Examples: Cantor set: $(-U-)$
Von Koch: $30°$ $30°$ — U - open triangle.

Does not have to contain K!

Theorem. Let K be an attractor of OSC family $(T_1,\dots,T_k)$ of uniform contractions of $\mathbb{R}^d$, α - its self-similarity dimension. Then $0<m_\alpha(K)<\infty$ and Hdim K = Mdim $K=\alpha$.

Pf First, let us define a measure μ for a probability vector $(r_1^\alpha,\dots,r_k^\alpha)$, i.e. $\mu(T_\sigma K)=r_{\sigma_1}^\alpha\cdots r_{\sigma_n}^\alpha$.
We will show that μ is α-smooth, and, by MDP, $m_\alpha(K)>0$ (We already know that $m_\alpha(K)<\infty$).

Prelim on μ and Σ_ε

To this end, fix $\varepsilon>0$ and consider the set Σ_ε of all multi-indices σ which satisfy $r_{\sigma_1}\cdots r_{\sigma_n}<\varepsilon\leq r_{\sigma_1}\cdots r_{\sigma_{n-1}}$. Σ_ε is an example of a Stopped set: for each infinite sequence $(\sigma_1,\dots\sigma_n,\dots)$ we keep adding elements till certain condition is satisfied. Observe that for each stopped set Σ, $(T_\sigma K)_{\sigma\in\Sigma}$ form a covering of K. Also, by induction the size of Σ, and the fact that $\sum r_i^\alpha=1$ and $\sum r_i^\alpha\mu T_i^{-1}=\mu$, one easily see that $\sum_{\sigma\in\Sigma}r_{\sigma_1}^\alpha\cdots r_{\sigma_n}^\alpha=1$ and $\mu=\sum_{\sigma\in\Sigma}r_{\sigma_1}^\alpha\cdots r_{\sigma_n}^\alpha\mu T_\sigma^{-1}$.
Returning to Σ_ε, observe that
$\min r_i\,\varepsilon\operatorname{diam}K<\operatorname{diam}T_\sigma K\leq r_{\sigma_1}\cdots r_{\sigma_n}\operatorname{diam}K\leq\varepsilon\operatorname{diam}K$

Using OSC to establish lower bound.

Now observe that if V - open set from OSC, then $\bar V\supset\bigcup T_i(\bar V)$, so $K\subset\bar V$ (T_i - uniform contractions of complete $\bar V$)
Let V contain some ball of radius a.
Then

$\{ \sigma V, \sigma \in \Sigma_\varepsilon \}$ are disjoint, each contain a ball of radius $a\varepsilon(\min r_i)$
Let us pick any ball $B(x,\varepsilon)$, $x \in K$. Then
if for some x, $\{\sigma \in \Sigma_\varepsilon, T_\sigma V \cap B(x,\varepsilon) \neq \emptyset\} = \{\sigma \in \Sigma_\varepsilon : T_\sigma V \cap B(x,\varepsilon) \neq \emptyset\}$
$T_\sigma V \subset B(x, \varepsilon(1+\mathrm{diam}(V)))$, and $\mathrm{Vol}(T_\sigma V) \geq c(d, a, \min r_i) \cdot \varepsilon^d$.
But $T_{\sigma_1} V \cap T_{\sigma_2} V = \emptyset$ if $\sigma_1, \sigma_2 \in \Sigma_\varepsilon$. Thus
$\#\{\sigma : T_\sigma V \cap B(x,\varepsilon) \neq \emptyset\} \cdot c(d, a, \min r_i)\varepsilon^d \leq \mathrm{Vol}(B(x, \varepsilon(1+\mathrm{diam}\, V)) \leq c'\varepsilon^d$
so $\#\{\sigma : \dots\} \leq C_2$.
so $m(B(x,\varepsilon)) = \sum_{\sigma \in \Sigma_\varepsilon} (r_{\sigma_1} \dots r_{\sigma_n})^\alpha m\left((T_{\sigma_1}^{-1} \circ \dots \circ T_{\sigma_n}^{-1})(B(x,\varepsilon))\right) \leq \varepsilon^\alpha \# \{\sigma : \dots\} \leq C\varepsilon^\alpha$.
so, by Mass Distribution Principle, $m_\alpha(K) > 0$.

Using Σ_α to establish upper bound.

Now let us prove that $\overline{M\dim} K \leq \alpha$.
The sets
$\{T_\sigma K, \sigma \in \Sigma_\varepsilon\}$ form a cover of K, by sets of diam $\leq \varepsilon\,\mathrm{diam}\,K$
so $N(\varepsilon\,\mathrm{diam}\,K; K) \leq \# \Sigma_\varepsilon$. But
$1 = \sum_{\sigma \in \Sigma_\varepsilon} (r_{\sigma_1} \dots r_{\sigma_n})^\alpha \geq (\varepsilon \min r_i)^\alpha \# \Sigma_\varepsilon$, so
$\# \Sigma_\varepsilon \leq (\varepsilon \min r_i)^{-\alpha}$, thus
$\overline{M\dim}\, K = \overline{\lim} \dfrac{\log N(K, \varepsilon\,\mathrm{diam}\,K)}{\log(1/\varepsilon\,\mathrm{diam}\,K)} \leq \alpha$
It follows from (1)
that $\alpha = H\dim K \leq \underline{M\dim}\, K \leq \overline{M\dim}\, K \leq \alpha$ ∎

Remark. OSC is necessary: for an attractor K $m_\alpha(K) > 0 \Rightarrow$ OSC ! (Schief, 1994)

Remark/Lemma Even without OSC, $H\dim K = M\dim K$.

Pf. Take V-open, $K \subset V$, $\mathrm{diam}\, V < 2\,\mathrm{diam}\, K$.
Any $x \in K$ is $\cap T_{\sigma|[1,n]} K$ for some (possibly non-unique) multi-index σ.
For $\varepsilon < \frac{\mathrm{dist}(K, \partial V)}{2}$, pick $n(\varepsilon)$: $T_{\sigma^{n(\varepsilon)}}(V) \subset B(x,\varepsilon)$,
$T_{\sigma^{n(\varepsilon)-1}}(V) \not\subset B(x,\varepsilon)$. Then $T_{\sigma^{n(\varepsilon)}} K \subset B(x,\varepsilon)$,
$\mathrm{diam}\; T_{\sigma^{n(\varepsilon)}} K \geq \dfrac{\mathrm{diam}\, T_{\sigma^{n(\varepsilon)-1}}(V)}{2\, r_{max}} \geq \dfrac{\varepsilon}{2r_{max}}$ and $r_{\sigma_1} \dots r_{\sigma_{n(\varepsilon)}} \geq \dfrac{\varepsilon}{2r_{max}}$
Now let $D < \underline{M\dim}\, K$, and pick ε, so that
$P(\varepsilon, K) \geq \left(\frac{1}{\varepsilon}\right)^D$. Then $(B(x_j,\varepsilon))_{j=1}^{P(\varepsilon,K)}$ are disjoint, then $T_{\sigma^{n(\varepsilon)}}(V) \subset V$, disjoint, there $P(\varepsilon,K)$ of them, the $h(\varepsilon)$ for x_j
with ratios at least $\frac{\varepsilon}{2r_{max}}$. Thus, for the attractor of this O.S.C. system K_1 we have $K_1 \subset K$, $H\dim K_1$ = self-similarity dimension of $K_1 \geq \dfrac{\log P(\varepsilon,K)}{\log\frac{1}{\varepsilon} + \log\frac{1}{2r_{max}}} \geq D \dfrac{1}{1 + \frac{\log\frac{\varepsilon}{2r_{max}}}{\log\frac{1}{\varepsilon}}}$ — the self-similarity dimension when all ratios are $\frac{\varepsilon}{2r_{max}}$.
Thus $H\dim K \geq H\dim K_1 \geq D$ for any $D < \underline{M\dim}\, K$, so $H\dim K \geq \underline{M\dim}\, K$ ∎